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A non-iterative method for boundary-layer equations—Part II: Two-dimensional laminar and turbulent flows

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SUMMARY

A non-iterative method for non-linear parabolic partial-differential equations is described and applied to boundary-layer equations for two-dimensional laminar and turbulent flows. Comparison of calculated results indicates that the accuracy of this method is comparable to those obtained with an iterative method. Copyright © 2003 John Wiley & Sons, Ltd.

KEY WORDS: boundary-layer equation; laminar flow; turbulent flow; iterative method; noniterative method; nonlinear parabolic partial-differential equation

1. INTRODUCTION

It is well known that the boundary-layer equations for laminar and turbulent flows are parabolic and non-linear. Their solutions can be obtained in their non-linear form by using a shooting method or in linearized form by a finite-difference method [1]. The latter choice is more common and practical, especially in turbulent flows since a finite-difference method is more efficient than an integration method such as a shooting method.

It is also well known that the solutions of the boundary-layer equations in differential form are more time consuming than those based on integral form since in the latter case the solutions are obtained for ordinary differential equations. Even though significant advances have been made in the development of efficient and accurate numerical methods to solve the differential form of the boundary-layer equations, the computer times associated with them are still considerably more than those of integral methods.

The 'edge' the integral methods have over the differential methods becomes even more significant in methods which employ inviscid/viscous interaction techniques where the boundary-layer equations are solved several times for a given pressure distribution by making several sweeps on the body. In each sweep, the solutions at each streamwise location are obtained

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iteratively since the equations have been linearized. This procedure is not only applied on every streamwise station but it is also used in each sweep. It is clear that significant savings in computer time can result if the solutions are not iterated at each streamwise location without sacrificing accuracy.

In the present paper, we address this need and describe a new numerical method that will allow computer savings of 75% of a modern finite-difference method such as that described in Reference [2]. The method is based on a non-iterative scheme [3] and its accuracy is comparable to that resulting from an iterative scheme. In this paper, we describe this method for two-dimensional laminar and turbulent flows and compare its solutions with an iterative scheme such as that described in Reference [2].

To the authors' knowledge, this is the first paper that describes the solution of the boundary-layer equations, which are non-linear parabolic partial-differential equations, with a noniterative scheme. As such, it is a useful and efficient tool leading to considerable computer savings and will find applications in many engineering problems that require the solution of nonlinear parabolic partial differential equations.

2. BOUNDARY-LAYER EQUATIONS

The boundary-layer equations and their boundary conditions for two-dimensional laminar and turbulent flows are well known. With an eddy viscosity (ε_m) concept,

$$\varepsilon_m = \frac{(-\overline{u'v'})}{\partial u/\partial y} \tag{1}$$

they can be written as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2}$$

$$u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} = u_e\,\frac{du_e}{dx} + \frac{\partial}{\partial y}\left(b\,\frac{\partial u}{\partial y}\right) \tag{3}$$

$$y = 0,\;\; u = 0,\;\; v = 0;\;\; y \to \infty, u \to u_e(x) \tag{4}$$

where

$$b = \varepsilon_m + v$$

As discussed in Reference [2], it is more efficient to solve the boundary-layer equations in transformed variables. A convenient one is the Falkner–Skan transformation in which a similarity parameter η is defined by

$$\eta = \sqrt{\frac{u_e}{vx}}\,y \tag{5a}$$

and a dimensionless stream function $f(x, \eta)$ by

$$f(x,\eta) = \frac{\psi(x, y)}{\sqrt{u_e v x}} \tag{5b}$$

Here $\psi(x, y)$ is a dimensional stream function that satisfies equation (2),

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \tag{6}$$

With this transformation, Equations (2) to (4) can be written as

$$(bf'')' + \frac{m+1}{2} ff'' + m[1 - (f')^2] = x\left(f'\frac{\partial f'}{\partial x} - f''\frac{\partial f}{\partial x}\right) \tag{7}$$

$$\eta = 0, \quad f = f' = 0; \quad \eta = \eta_{\mathrm{e}}, \quad f' = 1 \tag{8}$$

Here m is a dimensionless pressure-gradient parameter

$$m = \frac{x}{u_e} \frac{\mathrm{d}u_e}{\mathrm{d}x} \tag{9}$$

3. ITERATIVE METHOD

Boundary-layer equations being parabolic in nature can be solved by using several numerical methods; of these, finite difference methods are at present the most common with Crank–Nicolson [2] and Keller's box [4] methods being the most popular ones. The latter has several advantages over the former and is considered here.

According to Keller's method, Equation (7) is first expressed as a first-order system by introducing new variables $u(x, \eta)$, and $v(x, \eta)$

$$f' = u \tag{10a}$$

$$u' = v \tag{10b}$$

so that Equation (7) becomes

$$(bv)' + \frac{m+1}{2} fv + m(1 - u^2) = x\left(u\frac{\partial u}{\partial x} - v\frac{\partial f}{\partial x}\right) \tag{10c}$$

The boundary conditions now become

$$\eta = 0, \ u = 0, \ f = 0; \ \eta = \eta_{\mathrm{e}}, \ u = 1 \tag{11}$$

3.1. Iterative method

To solve the system given by Equations (10) and (11) with Keller's box method, which is an iterative method, we consider the net rectangle shown in Figure 1

$$\begin{aligned} &x^0 = 0, \quad x^n = x^{n-1} + k_n, \quad n = 1, 2, \ldots, N \\ &\eta_0 = 0, \quad \eta_j = \eta_{j-1} + h_j, \quad j = 1, 2, \ldots, J; \quad \eta_J = \eta_e \end{aligned} \tag{12}$$

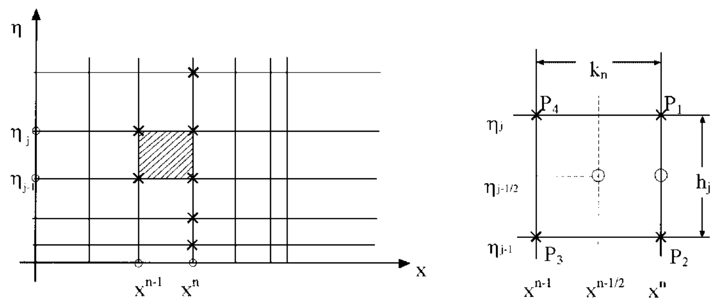

Figure 1. Net rectangle for distance approximations.

and write difference equations that are to approximate Equations (10). We write the finite difference approximations of Equations 10(a) and 10(b) at $(x^n, \eta_{j-1/2})$ using centred-difference derivatives.

$$\frac{f_j^n - f_{j-1}^n}{h_j} - \frac{u_j^n + u_{j-1}^n}{2} = 0 \tag{13a}$$

$$\frac{u_j^n - u_{j-1}^n}{h_j} - \frac{v_j^n + v_{j-1}^n}{2} = 0 \tag{13b}$$

Similarly, Equation (10c) is approximated at the midpoint $(x^{n-1/2}, \eta_{j-1/2})$ of the net rectangle $P_1P_2P_3P_4$,

$$\begin{aligned}
&\frac{1}{2}\left[\frac{(bv)_j^n - (bv)_{j-1}^n}{h_j} + \frac{(bv)_j^{n-1} - (bv)_{j-1}^{n-1}}{h_j}\right] + \frac{m^{n-1/2}+1}{4}\left[(fv)_{j-1/2}^n + (fv)_{j-1/2}^{n-1}\right] \\
&+ m^{n-1/2}\left(1 - \frac{((u^2)_{j-1/2}^n + (u^2)_{j-1/2}^{n-1})}{2}\right) \\
&= x^{n-1/2}\left[u_{j-1/2}^{n-1/2}\,\frac{u_{j-1/2}^n - u_{j-1/2}^{n-1}}{k_n} - v_{j-1/2}^{n-1/2}\,\frac{f_{j-1/2}^n - f_{j-1/2}^{n-1}}{k_n}\right]
\end{aligned} \tag{13c}$$

The boundary conditions, Equation (11) become

$$u_0^n = 0; \quad f_0^n = 0; \quad u_J^n = 1 \tag{14}$$

If we assume $f_j^{n-1}, u_j^{n-1}, v_j^{n-1}$ to be known for $0 \leqslant j \leqslant J$, then Equations (13) and (14) form a system of $3J+3$ equations for the solution of $3J+3$ unknowns f_j^n, u_j^n, v_j^n, $0 \leqslant j \leqslant J$. To solve this non-linear system, we use Newton's method and solve the resulting linear system with the block-elimination method described in detail in Reference [2].

3.2. Noniterative method

To solve the system given by Equations (10) and (11) with the non-iterative method, we first write Equation (10c) at $(x^{n-1/2}, \eta)$ as

$$\frac{((bv)')^n + ((bv)')^{n-1}}{2} + \frac{m^{n-1/2}+1}{4}(f^n v^{n-1} + f^{n-1}v^n) + m^{n-1/2}(1 - u^n u^{n-1})$$
$$= \frac{x^{n-1/2}}{2}\left(u^n \frac{\partial u^{n-1}}{\partial x} + u^{n-1}\frac{\partial u^n}{\partial x} - v^n \frac{\partial f^{n-1}}{\partial x} - v^{n-1}\frac{\partial f^n}{\partial x}\right) \tag{15}$$

Since

$$f^n = f^{n-1/2} + \frac{\partial f^{n-1/2}}{\partial x}\frac{k_n}{2} + \frac{1}{2!}\frac{\partial^2 f^{n-1/2}}{\partial x^2}\left(\frac{k_n}{2}\right)^2 + \mathrm{o}(k_n^3) \tag{16a}$$

and

$$f^{n-1} = f^{n-1/2} - \frac{\partial f^{n-1/2}}{\partial x}\frac{k_n}{2} + \frac{1}{2!}\frac{\partial^2 f^{n-1/2}}{\partial x^2}\left(\frac{k_n}{2}\right)^2 + \mathrm{o}(k_n^3) \tag{16b}$$

we can write

$$(fv)^{n-1/2} = \frac{(f^n v^{n-1} + f^{n-1}v^n)}{2} + \mathrm{o}(k_n^2) \tag{16c}$$

Similarly, we can represent the other terms in Equation (15) with second order approximations,

$$\begin{aligned}(u_x)_{j-1/2}^{n-1} &= a_1 u_{j-1/2}^{n-3} + a_2 u_{j-1/2}^{n-2} + a_3 u_{j-1/2}^{n-1}\\ (u_x)_{j-1/2}^{n} &= \tilde{a}_1 u_{j-1/2}^{n-2} + \tilde{a}_2 u_{j-1/2}^{n-1} + \tilde{a}_3 u_{j-1/2}^{n}\end{aligned} \tag{17}$$

where

$$\begin{aligned}
a_1 &= \frac{x_{n-1}-x_{n-2}}{(x_{n-2}-x_{n-3})(x_{n-1}-x_{n-3})}, \quad & a_2 &= -\frac{x_{n-1}-x_{n-3}}{(x_{n-2}-x_{n-3})(x_{n-1}-x_{n-2})}\\
a_3 &= \frac{2x_{n-1}-x_{n-2}-x_{n-3}}{(x_{n-1}-x_{n-2})(x_{n-1}-x_{n-3})}, \quad & \tilde{a}_1 &= \frac{x_n - x_{n-1}}{(x_{n-1}-x_{n-2})(x_n - x_{n-2})}\\
\tilde{a}_2 &= -\frac{x_n - x_{n-2}}{(x_{n-1}-x_{n-2})(x_n - x_{n-1})}, \quad & \tilde{a}_3 &= \frac{2x_n - x_{n-1} - x_{n-2}}{(x_n - x_{n-1})(x_n - x_{n-2})}
\end{aligned} \tag{18}$$

Equation (15) can then be written as

$$\frac{(bv)_j^n - (bv)_{j-1}^n}{h_j} + \frac{(bv)_j^{n-1} - (bv)_{j-1}^{n-1}}{h_j} + \frac{m^n+1}{2}(f_{j-1/2}^n\ v_{j-1/2}^{n-1} + v_{j-1/2}^n\ f_{j-1/2}^{n-1})$$
$$+2m^n(1 - u_{j-1/2}^n\ u_{j-1/2}^{n-1}) = x^{n-1/2}\begin{bmatrix} u_{j-1/2}^n(a_1 u_{j-1/2}^{n-3} + a_2 u_{j-1/2}^{n-2} + a_3 u_{j-1/2}^{n-1})\\ +u_{j-1/2}^{n-1}(\tilde{a}_1 u_{j-1/2}^{n-2} + \tilde{a}_2 u_{j-1/2}^{n-1} + \tilde{a}_3 u_{j-1/2}^{n})\\ -v_{j-1/2}^n(a_1 f_{j-1/2}^{n-3} + a_2 f_{j-1/2}^{n-2} + a_3 f_{j-1/2}^{n-1})\\ -v_{j-1/2}^{n-1}(\tilde{a}_1 f_{j-1/2}^{n-2} + \tilde{a}_2 f_{j-1/2}^{n-1} + \tilde{a}_3 f_{j-1/2}^{n})\end{bmatrix} \tag{19}$$

Table I. Comparison of calculated results for $k = 1/16$.

$\Delta x = 1/16$	Iterative method		Non-iterative method		$(F''_{\mathrm{w}})_{\mathrm{iter}} - (f''_{\mathrm{w}})_{\mathrm{non}}$	Relative
x	$(f''_{\mathrm{w}})_{\mathrm{iter}}$	Iter	$(f''_{\mathrm{w}})_{\mathrm{non}}$	Iter	Diff.	Diff.
0.25	2.80E − 01	3	2.80E − 01	1	1.00E − 04	3.57E − 04
0.3125	2.66E − 01	3	2.66E − 01	1	0.00E + 00	0.00E + 00
0.375	2.50E − 01	3	2.50E − 01	1	1.00E − 04	3.99E − 04
0.4375	2.34E − 01	3	2.34E − 01	1	0.00E + 00	0.00E + 00
0.5	2.18E − 01	3	2.18E − 01	1	1.00E − 04	4.60E − 04
0.5625	2.00E − 01	3	2.00E − 01	1	0.00E + 00	0.00E + 00
0.625	1.81E − 01	3	1.80E − 01	1	2.00E − 04	1.11E − 03
0.6875	1.60E − 01	3	1.60E − 01	1	2.00E − 04	1.25E − 03
0.75	1.37E − 01	3	1.37E − 01	1	3.00E − 04	2.19E − 03
0.8125	1.11E − 01	3	1.11E − 01	1	5.00E − 04	4.50E − 03
0.875	8.02E − 02	3	7.95E − 02	1	7.10E − 04	8.85E − 03
0.9375	3.95E − 02	4	3.88E − 02	1	6.30E − 04	1.60E − 02
0.9425	3.42E − 02	3	3.25E − 02	1	1.68E − 03	4.92E − 02
0.9475	2.83E − 02	3	2.66E − 02	1	1.75E − 03	6.18E − 02
0.9525	2.18E − 02	3	1.93E − 02	1	2.48E − 03	1.14E − 01
0.9575	1.32E − 02	3	9.44E − 03	1	3.74E − 03	2.84E − 01
	Total iteration	49		16		

If we assume f_j^n, u_j^n, v_j^n to be unknowns for $0 \leqslant j \leqslant J$, then Equations (13a), (13b) and (19) form a linear algebraic system of $3J+3$ equations that can be solved with the same procedure used for the iterative method [2].

4. ACCURACY OF THE NONITERATIVE METHOD

To evaluate the accuracy of the noniterative method, we have performed calculations with both methods for laminar and turbulent flows as discussed below.

4.1. Laminar flow

We consider Howarth's flow for which the inviscid velocity distribution is given by

$$u_e(x) = 1 - \frac{1}{8}x$$

and perform calculations with fixed uniform spacing in the x-direction with $k = 1/16$ and $1/32$.

This flow has a separation at $x = 0.960$ which is generally obtained by extrapolation. For calculations to proceed as close to this location, much finer spacings are needed around $x \approx 0.95$. So we expect the calculations to break down with course spacing earlier than those with fine spacing.

Tables I and II show the calculated wall-shear parameter for $k = 1/16$ and $1/32$, respectively. A choice of $k = 1/16$ yields 16 x-stations, the last being at $x = 0.9375$. This is a rather severe test case because, since the iterative method uses Newton's method, a course grid will increase the number of iterations.

Table II. Comparison of calculated results for $k = 1/16$.

$\Delta x = 1/32$	Iterative method		Noniterative method		$(f''_w)_{iter} - (f''_w)_{non}$	Relative
x	$(f''_w)_{iter}$	Iter	$(f''_w)_{non}$	Iter	Diff.	Diff.
0.125	3.07E − 01	2	3.07E − 01	1	0.00E + 00	0.00E + 00
0.1563	3.01E − 01	2	3.01E − 01	1	0.00E + 00	0.00E + 00
0.1875	2.94E − 01	2	2.94E − 01	1	0.00E + 00	0.00E + 00
0.2188	2.87E − 01	2	2.87E − 01	1	0.00E + 00	0.00E + 00
0.25	2.80E − 01	2	2.80E − 01	1	−1.00E − 04	−3.57E − 04
0.2813	2.73E − 01	2	2.73E − 01	1	−1.00E − 04	−3.67E − 04
0.3125	2.66E − 01	2	2.66E − 01	1	0.00E + 00	0.00E + 00
0.3438	2.58E − 01	2	2.58E − 01	1	0.00E + 00	0.00E + 00
0.375	2.50E − 01	2	2.50E − 01	1	0.00E + 00	0.00E + 00
0.4063	2.43E − 01	2	2.43E − 01	1	0.00E + 00	0.00E + 00
0.4375	2.35E − 01	2	2.35E − 01	1	0.00E + 00	0.00E + 00
0.4688	2.26E − 01	2	2.26E − 01	1	0.00E + 00	0.00E + 00
0.5	2.18E − 01	3	2.18E − 01	1	1.00E − 04	4.60E − 04
0.5313	2.09E − 01	3	2.09E − 01	1	0.00E + 00	0.00E + 00
0.5625	2.00E − 01	3	2.00E − 01	1	1.00E − 04	5.01E − 04
0.5938	1.90E − 01	3	1.90E − 01	1	0.00E + 00	0.00E + 00
0.625	1.81E − 01	3	1.81E − 01	1	1.00E − 04	5.54E − 04
0.6563	1.70E − 01	3	1.70E − 01	1	1.00E − 04	5.87E − 04
0.6875	1.60E − 01	3	1.60E − 01	1	1.00E − 04	6.26E − 04
0.7188	1.49E − 01	3	1.49E − 01	1	2.00E − 04	1.35E − 03
0.75	1.37E − 01	3	1.37E − 01	1	2.00E − 04	1.46E − 03
0.7813	1.24E − 01	3	1.24E − 01	1	2.00E − 04	1.61E − 03
0.8125	1.11E − 01	3	1.11E − 01	1	3.00E − 04	2.70E − 03
0.8438	9.65E − 02	3	9.61E − 02	1	3.50E − 04	3.63E − 03
0.875	8.02E − 02	3	7.98E − 02	1	4.90E − 04	6.11E − 03
0.9063	6.12E − 02	3	6.05E − 02	1	6.90E − 04	1.13E − 02
0.9375	3.83E − 02	4	3.73E − 02	1	1.00E − 03	2.61E − 02
0.9425	3.31E − 02	3	3.14E − 02	1	1.67E − 03	5.05E − 02
0.9475	2.71E − 02	3	2.52E − 02	1	1.91E − 03	7.05E − 02
0.9525	2.04E − 02	3	1.79E − 02	1	2.58E − 03	1.26E − 01
0.9575	1.12E − 02	4	6.88E − 03	1	4.30E − 03	3.84E − 01
	Total	83		31		

Table I shows the results for $k = 1/16$. Since the non-iterative method requires that solutions to previous three x-stations are known, the comparison of the calculated results in Table I begin at the fourth x-station, $x = 0.25$. Also, this table contains additional x-stations between $x = 0.9375$ and 0.96 with $\Delta x = 0.005$, due to flow separation.

As can be seen, the solutions obtained from the noniterative method agree well with those obtained from the iterative method, especially for values of x away from the separation location. As we approach the separation location, say $x \approx 0.90$, the difference between the two solutions increases gradually. Within a plotting accuracy shown in Figure 2, the agreement is very good, including the location of flow separation point. In addition, the comparison shows that while the iterative method requires a total of 49 iterations, the noniterative method requires only 16 iterations, leading to computational work reduction by 70%. The convergence criteria on the change of f''_{wall} is 10^{-4}.

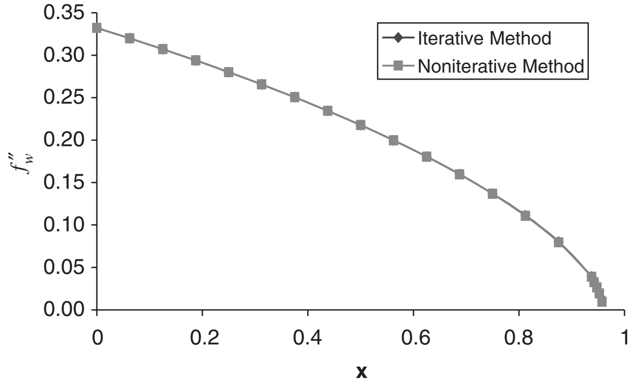

Figure 2. Comparison of calculated wall shear values with both methods for $k=1/16$.

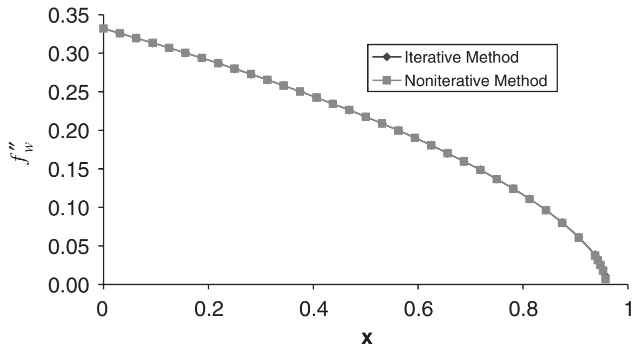

Figure 3. Comparison of calculated wall shear values with both methods for $k=1/32$.

Table II and Figure 3 show the results for $k=1/32$ with conclusions essentially the same as those obtained with $k=1/16$. Again, both results agree very well until $x\approx 0.90$ and begin to differ with increasing x. The computational work reduction of the non-iterative method (83 iterations vs 31 iterations) is again good, but this time it is around 60%.

4.2. Turbulent flow

To compare the predictions of the non-iterative method with the iterative method, we have considered a turbulent flow with zero pressure gradient with eddy viscosity formulation given by the Cebeci–Smith model [4]. This formulation treats a turbulent boundary layer as a composite layer with inner and outer regions and uses separate eddy-viscosity formulas in each region, expressing b in Equation (7) as a function of f'' and f. As a result, there are several choices for writing finite-difference approximations to $(bf'')'$ in the non-iterative method. Here we assume that b is given with values at x_{n-1}, thus making this term known prior to performing calculations at $x=x_n$. Other choices are being investigated and will be reported later.

Table III. Comparison of calculated results for turbulent flow on a flat plate.

	Iterative method		Noniterative method		$(f''_w)_{iter} - (f''_w)_{non}$	Relative
x	$(f''_w)_{iter}$	Iter	$(f''_w)_{non}$	Iter	Diff.	Diff.
0.4	1.27E+00	5	1.33E+00	1	−5.50E − 02	−4.32E − 02
0.5	1.38E+00	2	1.39E+00	1	−9.00E − 03	−6.54E − 03
0.6	1.43E+00	3	1.46E+00	1	−2.10E − 02	−1.46E − 02
0.7	1.51E+00	2	1.52E+00	1	−2.00E − 03	−1.32E − 03
0.8	1.56E+00	2	1.58E+00	1	−1.50E − 02	−9.62E − 03
0.9	1.63E+00	2	1.63E+00	1	−2.00E − 03	−1.23E − 03
1	1.68E+00	2	1.69E+00	1	−1.30E − 02	−7.75E − 03
2	2.04E+00	2	2.08E+00	1	−3.40E − 02	−1.67E − 02
3	2.40E+00	6	2.43E+00	1	−2.80E − 02	−1.17E − 02
4	2.65E+00	2	2.67E+00	1	−1.60E − 02	−6.03E − 03
5	2.88E+00	2	2.93E+00	1	−5.30E − 02	−1.84E − 02
6	3.12E+00	3	3.10E+00	1	2.00E − 02	6.42E − 03
7	3.29E+00	2	3.29E+00	1	3.00E − 03	9.11E − 04
8	3.45E+00	2	3.44E+00	1	7.00E − 03	2.03E − 03
9	3.61E+00	2	3.60E+00	1	1.40E − 02	3.88E − 03
10	3.75E+00	2	3.76E+00	1	−1.30E − 02	−3.47E − 03
	Total iterations	41		16		

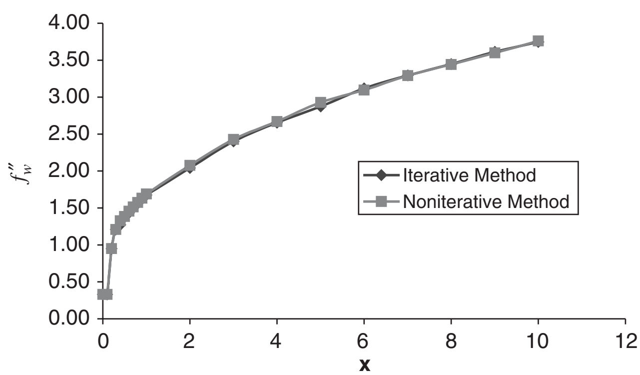

Figure 4. Comparison of calculated wall shear values for turbulent flow on a flat plate.

Table III and Figure 4 show the results for a turbulent flow on a flat plate with unit Reynolds number corresponding to 10^6. The calculations with the iterative method started as laminar at $x = 0$ and transition location was specified at $x = 0.2$ and were continued with $k = 0.1$ in [0, 1] and $k = 1$ in $[1, 10]$. The calculations with the non-iterative method started at the fourth x-station, $x = 0.40$. As can be seen, the predictions of the noniterative method are in very good agreement with those obtained with the iterative method. The efficiency of the noniterative method is similar to the efficiency in laminar flows: whereas the iterative method requires 41 iterations, the noniterative method requires only 16.

Table IV. Comparison of calculated results for turbulent flow on a flat plate.

	Iterative method		Noniterative method		$(f''_w)_{iter} - (f''_w)_{non}$	Relative
x	$(f''_w)_{iter}$	Iter	$(f''_w)_{non}$	Iter	Diff.	Diff.
0.4	1.24E+00	1	1.24E+00	1	4.00E − 03	3.22E − 03
0.5	1.32E+00	2	1.33E+00	1	−7.00E − 03	−5.29E − 03
0.6	1.40E+00	2	1.40E+00	1	−1.00E − 03	−7.15E − 04
0.7	1.47E+00	2	1.48E+00	1	−2.00E − 03	−1.36E − 03
0.8	1.53E+00	2	1.55E+00	1	−1.30E − 02	−8.47E − 03
0.9	1.60E+00	1	1.60E+00	1	0.00E + 00	0.00E + 00
1	1.66E+00	2	1.66E+00	1	2.00E − 03	−1.20E − 03
2.5	2.17E+00	2	2.17E+00	1	−2.00E − 03	−9.21E − 04
5	2.90E+00	1	2.87E+00	1	2.20E − 02	7.60E − 03
7.5	3.33E+00	2	3.34E+00	1	−1.30E − 02	−3.91E − 03
10	3.70E+00	2	3.83E+00	1	−1.29E − 01	−3.48E − 02
	Total iterations	19		11		

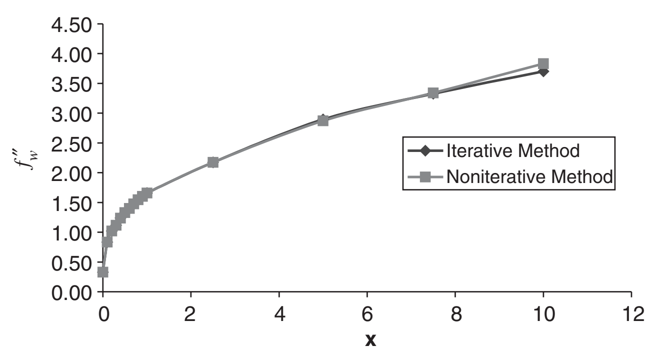

Figure 5. Comparison of calculated wall shear values for turbulent flow on a flat plate.

Table IV and Figure 5 present the results similar to those in Table III and Figure 4 except k in $[1, 10]$ is 2.5. As can be seen, the predictions of the noniterative method differ from those predicted with the iterative method. The reason for this is clearly the way b is calculated. Other choices for handling the b-term are in progress and will be reported later.

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